Resummation of cactus diagrams in lattice QCD

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We show how to perform a resummation, to all orders in perturbation theory, of a certain class of gauge-invariant diagrams in lattice QCD. These diagrams are often largely responsible for lattice artifacts. Our resummation leads to an improved perturbative expansion. Applied to a number of cases of interest, this expansion yields results remarkably close to corresponding nonperturbative estimates.
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I. INTRODUCTION

Ever since the earliest days of lattice field theory, one problem present in most numerical simulations has been the calculation of corrections induced by renormalization on Monte Carlo results. Although this notorious problem has not as yet been adequately dealt with, several methods have been used to address it: To begin with, perturbation theory provides in principle a methodical means of calculating, order by order in the coupling, renormalization functions, operator mixing coefficients, etc. Its drawbacks lie in its asymptotic nature and that it is a formidable task on the lattice, which places severe limitations on the order to which it can be carried out; indeed, at present, exact calculations in perturbative lattice QCD reach only two loops (for two-point diagrams) [1–3] and three loops (for vacuum diagrams) [4]. Various nonperturbative, numerical approaches to renormalization functions have also been devised and there has been recent progress both in their range of applicability and in their precision [5–7]. Finally, much effort has also gone in studying improved actions (which may, among other advantages, show improved renormalization behavior) [8,9] and improved or boosted perturbation theory [10].

In this paper, we present an improvement of lattice perturbation theory, which results from a resummation to all orders of a certain class of diagrams, dubbed ''cactus'' diagrams. Briefly stated, these are tadpole diagrams which become disconnected if any one of their vertices is removed (see Fig. 1). Our original motivation was the well-known observation of ''tadpole dominance'' in lattice perturbation theory (see, e.g., [11]). This observation must clearly be taken with a grain of salt: One-sided inclusion of tadpoles can ruin desirable partial cancellations between tadpole and nontadpole diagrams; worse, their contribution is gauge dependent. The class of terms we propose to resum circumvents the latter objection since, as we shall see, it is gauge invariant; it also overcomes the former objection in known cases.

Cactus resummation may be applied either to bare quantities or to quantities which have been calculated to a given order in perturbation theory; thus contributions which are not included in the resummation can be reintroduced in a systematic manner.

In Sec. II we present our calculation, leading to expressions for a dressed propagator and dressed vertices of interest; some derivations and technical details are relegated to the Appendixes. In Sec. III, we proceed to use these expressions to calculate various renormalization functions and compare our results with other methods: We find a remarkable improvement in many cases.

II. CALCULATION

A. Dressed propagator

Consider the standard Wilson action for SU(N) lattice gauge fields:

$$S=\frac{1}{g_0^2}\sum_{x,\mu\nu} \text{Re tr}(1-U^{\square}_{x,\mu\nu}). \qquad (1)$$

$U^{\square}_{x,\mu\nu}$ is the usual product of link variables around a plaquette in the μ-ν plane with the origin at x; in standard notation it reads

$$U^{\square}_{x,\mu\nu}=e^{ig_0A_{x,\mu}}e^{ig_0A_{x+\mu,\nu}}e^{-ig_0A_{x+\nu,\mu}}e^{-ig_0A_{x,\nu}},$$

$$A_{x,\mu}=A^a_{x,\mu}T^a. \qquad (2)$$

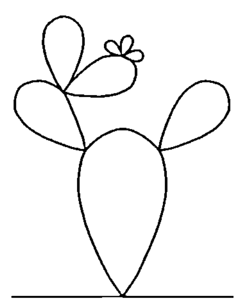

FIG. 1. A cactus.

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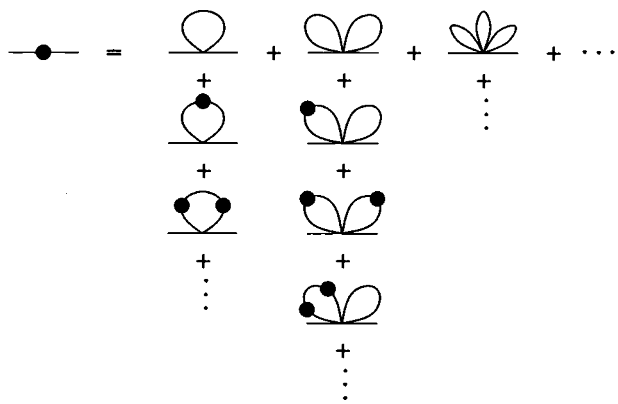
By the Baker-Campbell-Hausdorff (BCH) formula we have

$$U^{\square}_{x,\mu\nu}=\exp\{ig_0(A_{x,\mu}+A_{x+\mu,\nu}-A_{x+\nu,\mu}-A_{x,\nu})+\mathcal{O}(g_0^2)\}$$
$$\equiv\exp\{ig_0F^{(1)}_{x,\mu\nu}+ig_0^2F^{(2)}_{x,\mu\nu}+ig_0^3F^{(3)}_{x,\mu\nu}+\mathcal{O}(g_0^4)\}. \quad (3)$$

The diagrams which we propose to resum to all orders will be cactus diagrams made of vertices containing $F^{(1)}_{x,\mu\nu}$. Let us see how such diagrams will dress the gluon propagator; we write

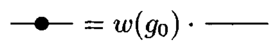

(4)

where the one-particle irreducible piece is given by the recursive equation

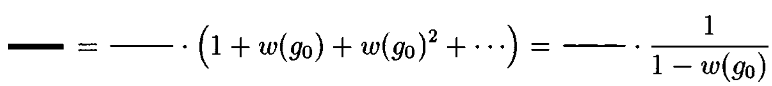

(5)

Now, the fact that the vertices involved in the above contain only $F^{(1)}_{x,\mu\nu}$ implies that the longitudinal parts of all propagators will always cancel. As we will see, this fact will lead to the result that the effect of dressing is the same in all covariant gauges. We will thus denote by a thick (thin) solid line the transverse dressed (bare) propagator.

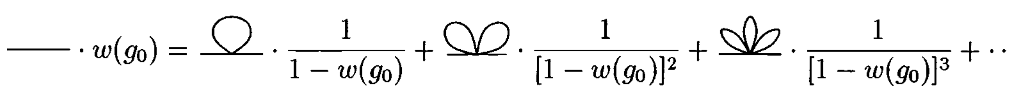
From Eq. (5) there follows

(6)


Indeed, the dressed propagator will become a multiple of the bare transverse one, where the factor $w(g_0)$ will depend on g_0 and N, but not on the momentum. Let us now turn the diagrammatic relations (4), (5) into an algebraic equation for $w(g_0)$; from Eq. (4) we have


(7)

and from Eq. (5) we find


(8)

It is crucial to verify at this stage that all diagrams contained above appear with the same combinatorial factors as in the ordinary perturbative expansion; this is indeed the case.

To proceed, we must evaluate the generic tadpole appearing in Eq. (8); this comes from an n-point vertex of the action, in which $n-2$ lines have been pairwise contracted. Before contraction, the vertex reads

$$-S\rightarrow\frac{1}{n!g_0^2}\sum_{x,\mu\nu}(ig_0)^n\mathrm{tr}\{(F^{(1)}_{x,\mu\nu})^n\}$$
$$=\frac{(ig_0)^n}{n!g_0^2}\sum_{x,\mu\nu}\int dq_1\cdots dq_n[\hat{q}_{1\mu}A^{a_1}_{\nu}(q_1)-\hat{q}_{1\nu}A^{a_1}_{\mu}(q_1)]\cdots[\hat{q}_{n\mu}A^{a_n}_{\nu}(q_n)-\hat{q}_{n\nu}A^{a_n}_{\mu}(q_n)]e^{i(q_1+\cdots+q_n)x}\mathrm{tr}\{T^{a_1}T^{a_2}\cdots T^{a_n}\} \quad (9)$$

$[\hat{q}_\mu=2\sin(q_\mu/2)]$. At contraction there will be $(n-2)/2$ loop integrations giving

$$\frac{1}{(2\pi)^4}\int d^4q\,\frac{2\hat{q}_\mu^2}{\hat{q}^2}=\frac{1}{2}. \quad (10)$$

For the contraction of the SU(N) generators we first define and evaluate $F(n;N)$, which is the sum over all complete pairwise contractions of $\mathrm{tr}\{T^{a_1}T^{a_2}\cdots T^{a_n}\}$:

$$F(n;N)=\frac{1}{2^{n/2}(n/2)!}\sum_{P\in S_n}\delta_{a_1a_2}\delta_{a_3a_4}\cdots\delta_{a_{n-1}a_n}\mathrm{tr}\{T^{P(a_1)}T^{P(a_2)}\cdots T^{P(a_n)}\} \tag{11}$$

[$F(2n+1;N)\equiv 0$; S_n is the permutation group of n objects]. In Appendix A we calculate the generating function of $F(n;N)$:

$$G(z;N)\equiv\sum_{n=0}^{\infty}\frac{z^n}{n!}F(n;N),\quad \text{whence}\ \ F(n;N)=\frac{d^n}{dz^n}G(z;N)\big|_{z=0}. \tag{12}$$

We find

$$G(z;N)=e^{z^2(N-1)/(4N)}L^1_{N-1}(-z^2/2) \tag{13}$$

(L^α_β are Laguerre polynomials). In the present case, two legs are left external, so that the color contraction gives

$$\frac{nF(n;N)}{2(N^2-1)}. \tag{14}$$

Substituting Eqs. (10) and (14) in Eq. (9), we obtain, for the tadpole,

$$\begin{aligned}&\frac{(ig_0)^n}{n!g_0^2}\sum_{x,\mu\nu}\int dq_1dq_2[\hat{q}_{1\mu}A^a_\nu(q_1)-\hat{q}_{1\nu}A^a_\mu(q_1)][\hat{q}_{2\mu}A^a_\nu(q_2)-\hat{q}_{2\nu}A^a_\mu(q_2)]e^{i(q_1+q_2)x}\frac{nF(n;N)}{2(N^2-1)}\left(\frac{1}{2}\right)^{(n-2)/2}\\&=\left|\frac{2(ig_0)^n}{(n-1)!g_0^2}\frac{1}{N^2-1}F(n;N)\left(\frac{1}{2}\right)^{(n-2)/2}\right..\end{aligned} \tag{15}$$

We can now sum up all terms in Eq. (8); we obtain

$$\begin{aligned}w(g_0)&=\sum_{n=4,6,8,\ldots}\frac{1}{[1-w(g_0)]^{(n-2)/2}}\frac{2(ig_0)^n}{(n-1)!g_0^2}\frac{1}{N^2-1}F(n;N)\left(\frac{1}{2}\right)^{(n-2)/2}\\&=\left\{\sum_{n=0}^{\infty}\frac{1}{[2-2w(g_0)]^{n/2}}\frac{(ig_0)^n}{n!}F(n+1;N)\right\}\frac{2(ig_0)}{g_0^2(N^2-1)}[2-2w(g_0)]^{1/2}+1.\end{aligned} \tag{16}$$

Comparing with the definition of $G(z;N)$, Eq. (12), we see that the expression in curly brackets above is simply $G'(z;N)$, the derivative of $G(z;N)$. Equation (16) now reads

$$zG'(z;N)\big|_{z=(ig_0)/[2-2w(g_0)]^{1/2}}=-\frac{g_0^2(N^2-1)}{4}. \tag{17}$$

From our result for $G(z;N)$, Eq. (13), we see that

$$zG'(z;N)=e^{z^2(N-1)/(4N)}\left[-\frac{N-1}{N}L^1_{N-1}\left(-\frac{z^2}{2}\right)-2L^2_{N-2}\left(-\frac{z^2}{2}\right)\right]\left(-\frac{z^2}{2}\right). \tag{18}$$

This allows us to make explicit Eq. (17):

$$ue^{-u(N-1)/(2N)}\left[\frac{N-1}{N}L^1_{N-1}(u)+2L^2_{N-2}(u)\right]=\frac{g_0^2(N^2-1)}{4}, \tag{19}$$

$$u(g_0)\equiv\frac{g_0^2}{4[1-w(g_0)]}.$$

Given g_0, N, this equation can be solved numerically for $u(g_0)$ and, subsequently, for $w(g_0)$. The region in g_0 for which a solution exists contains the whole range of physical interest. Indeed one finds a solution in the region $0\leqslant g_0^2\leqslant 16/3e^{1/2}\simeq 3.23$ for $N=2$ and $0\leqslant g_0^2\leqslant 1.558$ for $N=3$. In Figs. 2 and 3 we plot the left-hand side of Eq. (19) versus u, for SU(2) and SU(3), respectively.

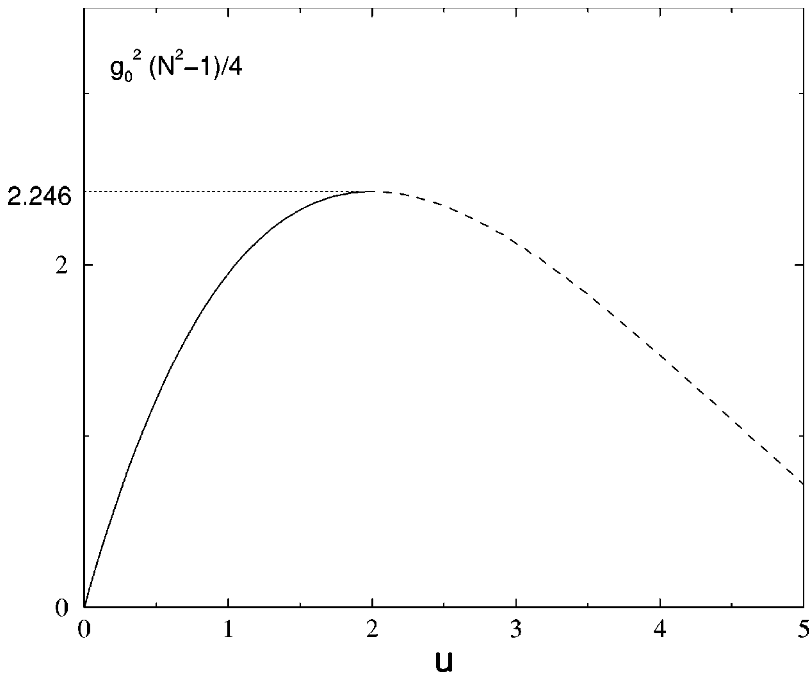


FIG. 2. Plot of the left-hand side of Eq. (19) versus u, for SU(2). The solid part of the curve identifies the interval of g_0 values for which a solution exists.

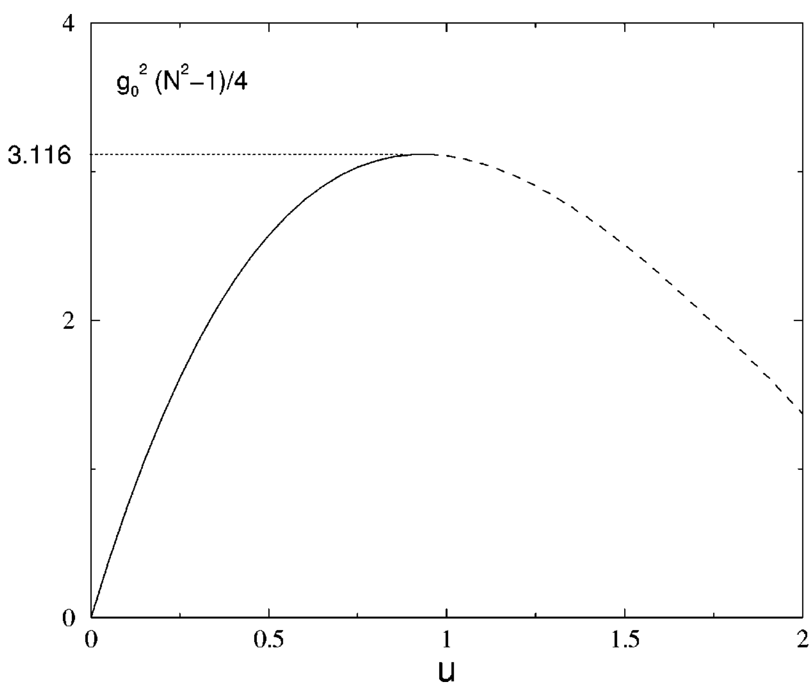
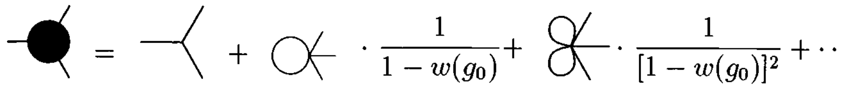


FIG. 3. As in Fig. 2, for the case of SU(3).

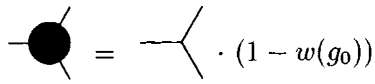
B. Vertices from the action

The three-point vertex of the action can be dressed to all orders in a manner similar to Eq. (5). We have



(20)

The calculation is described in Appendix B. The result turns out to be very simple:



(21)

where $w(g_0)$ is the quantity calculated previously.

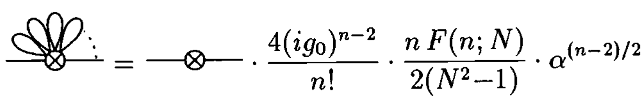
Vertices with more lines can be treated similarly; however, the dressed vertex in these cases is not merely a multiple of the bare one, which tends to complicate matters somewhat. Since we will not need such vertices for the numerical results of Sec. III, we only present some relevant formulas in Appendix C.

C. Other operators

Various lattice operators can be dressed by the same procedure. Let us take as an example a typical operator involving gluons:

$$\mathcal{O}=\sum \operatorname{tr}\{U_1 U_2 \cdots U_n\} \equiv \sum \operatorname{tr}\{e^{i g_0 Q}\}, \tag{22}$$

where the sum runs over the Lorentz indices involved. Using the first order BCH expansion for Q, we can write, once again for the two-point tadpole built out of an n-point vertex,



(23)

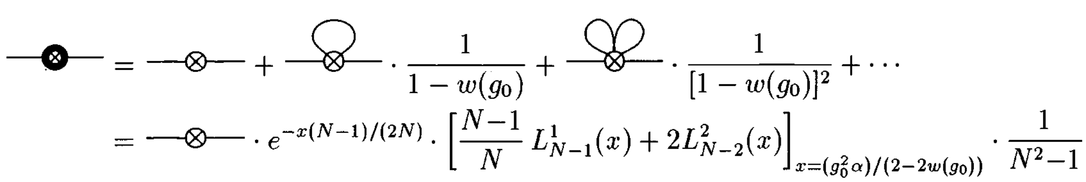
Here, α is the value of the one-loop momentum integration coming from the contraction of Q with itself; it is a pure number which depends on the operator under consideration. For example,

$$\mathcal{O}=S, \qquad \alpha=\frac{1}{2}, \quad \text{as before,} \tag{24}$$

$$\mathcal{O}=\sum_{\mu,\nu,\rho,\sigma} \varepsilon^{\mu\nu\rho\sigma} \operatorname{tr}\{U_{x,\mu\nu} U_{x,\rho\sigma}\}, \qquad \alpha=1, \tag{25}$$

$$\mathcal{O}=\sum_{\mu,\nu,\rho} \operatorname{tr}\{U_{x,\mu\nu}[U_{x,\nu\rho}, U_{x,\rho\mu}]\}, \qquad \alpha=\frac{3}{2}-\frac{3}{2(2\pi)^4}\int d^4q\, \frac{\hat{q}_\mu^2 \hat{q}_\nu^2}{\hat{q}^2} \simeq 0.85332, \tag{26}$$

and so on. The complete resummation of cactus diagrams then leads to



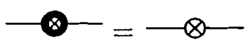
(27)

It turns out that three-point bare and dressed vertices are related by the same proportionality factor as the two-point vertices, Eq. (27).

For the topological charge density operator of Eq. (25) an alternative resummation is possible by using the BCH expansion as follows:

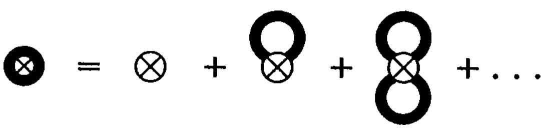
$$\sum_{\mu,\nu,\rho,\sigma} \varepsilon^{\mu\nu\rho\sigma} \mathrm{tr}\{U_{x,\mu\nu}U_{x,\rho\sigma}\}$$
$$= \sum_{\mu,\nu,\rho,\sigma} \varepsilon^{\mu\nu\rho\sigma} \mathrm{tr}\{\exp(ig_0 F_{x,\mu\nu})\exp(ig_0 F_{x,\rho\sigma})\}. \qquad (28)$$

Keeping the first order terms in $F_{x,\mu\nu}$ and $F_{x,\rho\sigma}$ the complete resummation leads to the simple result


$$\cdot\, [1-w(g_0)]^2 \qquad (29)$$

The square in the above expression can be traced to the fact that the operator is composed of two mutually orthogonal plaquettes.

Cactus resummation can be also used to estimate the perturbative vacuum expectation value of an operator:


(30)

This can be shown to equal


$$= G\left(\frac{i g_0 \alpha^{1/2}}{[1-w(g_0)]^{1/2}};\, N\right) \qquad (31)$$

D. Other representations

The calculation performed above can be generalized to encompass several other cases, e.g., operators involving higher representations for gluons. To illustrate this, we consider a class of variant actions proposed some time ago [12]:

$$S = \frac{\beta}{2}\sum_{x,\mu\nu}\left(1-\frac{1}{N}\,\mathrm{tr}U_{x,\mu\nu}\right) + \frac{\beta_A}{2}\sum_{x,\mu\nu}\left(1-\frac{1}{N^2-1}\,\mathrm{tr}_A U_{x,\mu\nu}\right). \qquad (32)$$

Here β and β_A are adjustable parameters and $\mathrm{tr}_A U_{x,\mu\nu}$ denotes the trace of a product of links in the adjoint representation, around a plaquette.

The calculation proceeds as before; the one new ingredient we need is $F_{\mathrm{Adj}}(n;N)$, the sum over all complete pairwise contractions of $\mathrm{tr}\{\mathcal{T}^{a_1}\mathcal{T}^{a_2}\cdots\mathcal{T}^{a_n}\}$ ($\mathcal{T}^a$ are generators in the adjoint representation). In Appendix D we compute $G_{\mathrm{Adj}}(z;N)$, the generating function for F_{Adj}. In terms of G_{Adj}, the equation for the factor $w_{\mathrm{var}}(g_0)$ which dresses the propagator now becomes

$$\left.\frac{\beta}{2N}\, zG'(z;N) + \frac{\beta_A}{2(N^2-1)}\, zG'_{\mathrm{Adj}}(z;N)\right|_{z=(ig_0)/[2-2w_{\mathrm{var}}(g_0)]^{1/2}} = -\frac{N^2-1}{4}, \qquad (33)$$

where

$$g_0^2 = \left[\frac{\beta}{2N} + \frac{\beta_A N}{N^2-1}\right]^{-1}. \qquad (34)$$

It is straightforward to solve Eq. (33) numerically for $w_{\mathrm{var}}(g_0)$.

We conclude this section by noting that the extension to vertices with fermions is immediate. First of all, vertices coming from the Wilson fermionic action stay unchanged, since their definition contains no plaquettes on which to apply the linear BCH formula. We will see how this affects corresponding renormalization functions in the next section. For more complicated fermionic vertices, such as those of the clover action, cactus resummation proceeds in precisely the same manner as Eqs. (23), (27).

III. SOME APPLICATIONS

In this section we apply the resummation of the cactus diagrams derived for the Wilson action to the calculation of the renormalization of some lattice operators. Approximate expressions for these renormalizations on the lattice are obtained by dressing the corresponding one-loop results. We will consider here operators whose anomalous dimensions are zero. A consistent, as well as physically motivated, means of implementing the cactus dressing is to apply it to the one-loop difference between lattice and continuum contributions that determine the renormalization, and not only to the lattice part. Cases with nonzero anomalous dimension can be dealt with in an analogous manner, by setting the scale $\mu = 1/a$ and dressing the finite renormalization coefficients as before.

As a first example we consider the calculation of the lattice renormalization $Z(g_0^2)$ of the topological charge density operator

$$Q(x) = -\frac{1}{2^4\times 32\pi^2}\sum_{\mu,\nu,\rho,\sigma=\pm 1}^{\pm 4} \varepsilon^{\mu\nu\rho\sigma}\mathrm{tr}\{U_{x,\mu\nu}U_{x,\rho\sigma}\}. \qquad (35)$$

$Z(g_0^2)$ is a finite function of g_0^2; it approaches 1 in the limit $g_0 \to 0$, and is much smaller than 1 in the region $g_0 \simeq 1$, where Monte Carlo simulations using the Wilson action are actually performed. A nonperturbative numerical calculation using the heating method [13] has produced the estimate [14,15]

$$Z(g_0^2 = 1) = 0.19(1) \quad \text{for SU(3)}. \qquad (36)$$

In this case few terms of the perturbation theory in g_0^2 can hardly provide an acceptable estimate of $Z(g_0^2)$ for $g_0^2 \simeq 1$ without some kind of resummation.

TABLE I. For the SU(2) lattice gauge theory we list the estimates of $Z(g_0^2)$ as obtained by the heating method [18] (h.m.), by the standard one-loop perturbative expansion (p.t.), and by cactus dressing, Eq. (38), of the one-loop calculation (d.p.t.).

$\beta \equiv 4/g_0^2$	h.m.	p.t.	d.p.t.
2.45	0.20(2)	0.125	0.219
2.5	0.22(1)	0.142	0.233
2.6	0.25(2)	0.175	0.259
2.8	0.32(2)	0.234	0.305
3.0	0.33(2)	0.285	0.347

In perturbation theory $Z(g_0^2)$ has been computed to $O(g_0^2)$ [16] with the result

$$Z(g_0^2)=1+z_1 g_0^2+O(g_0^4),$$
$$z_1=N\left(\frac{1}{4N^2}-\frac{1}{8}-\frac{1}{2\pi^2}-0.15493\right). \tag{37}$$

Numerically $z_1\simeq-0.908$ for SU(3) and $z_1\simeq-0.536$ for SU(2). So perturbation theory to $O(g_0^2)$ would give $Z(g_0^2)\simeq 0.092$ for SU(3), which is very far from its actual value, Eq. (36). In order to obtain a better approximation, we perform a cactus dressing of the one-loop calculation. The tree order is dressed by mere use of Eq. (29). One can now dress the one-loop contributions (for details of the standard perturbative calculation see Refs. [16, 17]). Using Eqs. (7), (21), and (29), and a simple combinatorial counting applied to the diagrams contributing to $Z(g_0^2)$, one arrives at the expression

$$Z(g_0^2)\approx[1-w(g_0)]^2+[1-w(g_0)]\left(z_1+\frac{2N^2-3}{12N}\right)g_0^2. \tag{38}$$

The quantity $(2N^2-3)/12N$ must be added to z_1 to avoid double counting, since such a contribution is already incorporated in the dressed tree-order approximation. Solving Eq. (19) for $N=3$ and $g_0=1$ one finds

$$1-w(g_0=1)=0.749775. \tag{39}$$

Thus from Eq. (38) one obtains $Z(g_0^2=1)\simeq 0.193$, which compares very well with the numerical result (36). Further confirmation of the validity of the approximation (38) comes from a comparison with available data for SU(2) in the range $2.45\leqslant\beta\leqslant 3.0$ ($\beta=4/g_0^2$) obtained by the heating method [18], as shown in Table I. The agreement is remarkable.

We wish to point out that other improvement recipes, such as those proposed in Ref. [10], consisting in a redefinition of the bare coupling, do not help in this case. For example one recipe entails the use of

$$\tilde{g}^2=\frac{g_0^2}{\frac{1}{3}\langle \mathrm{Tr} U^{\square}\rangle} \tag{40}$$

(where $U^{\square}$ is the plaquette) as the expansion parameter for $N=3$. In many cases this recipe represents an improvement. However, substituting the value of $\tilde{g}^2$ corresponding to $g_0^2=1$, i.e., $\tilde{g}^2\simeq 1.68$, in Eq. (37), one would obtain $Z(g_0^2=1)\simeq-0.54$, which is much worse than the plain one-loop approximation. Similarly, a change of coupling constant and momentum scale, in the manner of Lepage and Mackenzie [10], also leads to a wider discrepancy in this case: indeed, the corresponding value of $\alpha(q^*)$ (defined in [10]) turns out to be too large.

One can also apply cactus resummation to the lattice renormalization of fermionic operators. Let us consider the local nonsinglet vector and axial currents $V_\mu^a=\bar{\psi}\lambda^a\gamma_\mu\psi$ and $A_\mu^a=\bar{\psi}\lambda^a\gamma_\mu\gamma_5\psi$. The lattice renormalizations of these operators, $Z_V(g_0^2)$ and $Z_A(g_0^2)$, respectively, are again finite functions of g_0. In perturbation theory and for SU(3) one has [19]

$$Z_{V,A}=1+z_{V,A}g_0^2+O(g_0^4), \tag{41}$$

where $z_V\simeq-0.17$ and $z_A\simeq-0.13$. Thus, at $g_0^2=1$ one-loop perturbation theory gives $Z_V(g_0^2=1)\simeq 0.83$ and $Z_A(g_0^2=1)\simeq 0.87$. For these fermionic operators, one may use cactus resummation to dress the gluon propagators appearing in the diagrams contributing to one-loop order, according to Eq. (7). This procedure leads to

$$Z_{V,A}\approx 1+z_{V,A}\,\frac{g_0^2}{1-w(g_0)}. \tag{42}$$

At $g_0^2=1$, this gives $Z_V\simeq 0.77$ and $Z_A\simeq 0.83$. One may compare these numbers with those obtained in nonperturbative calculations based on Ward identities [5]. The only limitation of this method is due to scaling corrections, which turn out to be rather large at $g_0^2\simeq 1$ in the case of the Wilson lattice formulation. Depending on the matrix element one looks at, at $g_0^2=1$ one finds values ranging from 0.57 to 0.79 for Z_V and from 0.72 to 0.85 for Z_A [20–22] (see also Ref. [23] for a review of these results). Other methods of improvement (see, e.g., [10], and also [23] for a partial review), using various boosting procedures, result in numbers ranging from 0.63 to 0.71 for Z_V and from 0.72 to 0.77 for Z_A. Hence, a conclusive comparison is not possible in these cases. A better numerical situation occurs when one considers the clover action [8], for which scaling corrections are largely reduced in the region where Monte Carlo simulations are performed and precise estimates can be obtained using the Ward identities (see Ref. [23] and references therein). An application of our cactus resummation to the clover action would require the dressing of the new fermion-gluon three-point vertex. This point is under investigation.

In conclusion, the above examples show that the resummation of cactus diagrams leads to a general improvement in the evaluation of lattice renormalizations based on perturbation theory. A combination of this method with improved actions is expected to give a reliable evaluation of renormalization functions, which can complement corresponding nonperturbative estimates. We hope to return to this issue in a future publication.

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APPENDIX A: CALCULATION OF $G(z;N)$

We wish to calculate $F(n;N)$, the sum over all complete pairwise contractions of $\mathrm{tr}\{T^{a_1}T^{a_2}\cdots T^{a_n}\}$. For even n, $F(n;N)$ is defined by

$$F(n;N)=\frac{1}{2^{n/2}(n/2)!}\sum_{P\in S_n}\delta_{a_1a_2}\delta_{a_3a_4}\cdots\delta_{a_{n-1}a_n}\mathrm{tr}\{T^{P(a_1)}T^{P(a_2)}\cdots T^{P(a_n)}\} \tag{A1}$$

$[F(2n+1;N)\equiv 0]$. S_n is the permutation group of n objects, and T^a are an orthonormal basis for su(N) in the fundamental representation, $\mathrm{tr}\{T^aT^b\}=\frac{1}{2}\delta^{ab}$.

We define

$$M\equiv\theta^aT^a,\quad \theta^2=\theta^a\theta^a\quad (a=1,\ldots,N^2-1),\quad \theta^a\in\mathbb{R}. \tag{A2}$$

Then $F(n;N)$ can be written as

$$F(n;N)=\frac{1}{\mathcal{N}}\int\prod_a d\theta^a e^{-\theta^2/2}\mathrm{tr}\{M^n\}=\frac{1}{\mathcal{N}}\int[dM]e^{-M^2}\mathrm{tr}\{M^n\},\quad \mathcal{N}=\int[dM]e^{-M^2}. \tag{A3}$$

[The normalization $\mathcal{N}$ is redefined below as necessary, to ensure that $F(0;N)=N$ remains valid.] By definition, $[dM]=\Pi_a d\theta^a$ is the integration measure over traceless Hermitian matrices. When the integrand is invariant under similarity transformations, as is our case, ''angular'' integrations can be performed, leaving behind an integral over the N eigenvalues λ_i [24]:

$$F(n;N)=\frac{1}{\mathcal{N}}\int\left(\prod_i d\lambda_i\right)\left[\prod_{i<j}(\lambda_i-\lambda_j)^2\right]\delta\left(\sum_i\lambda_i\right)\exp\left(-\sum_i\lambda_i^2\right)\left(\sum_i\lambda_i^n\right). \tag{A4}$$

At this stage, it is convenient to introduce the generating function for $F(n;N)$:

$$G(z;N)\equiv\sum_{n=0}^{\infty}\frac{z^n}{n!}F(n;N),\quad F(n;N)=\frac{d^n}{dz^n}G(z;N)\Big|_{z=0}. \tag{A5}$$

By Eq. (A4) we have

$$\begin{aligned}G(z;N)&=\sum_i\int\frac{\Pi_jd\lambda_j}{\mathcal{N}}\left[\prod_{k<l}(\lambda_k-\lambda_l)^2\right]\delta\left(\sum_m\lambda_m\right)\exp\left(-\sum_n\lambda_n^2+z\lambda_i\right)\\&=N\int\frac{\Pi_jd\lambda_j}{\mathcal{N}}\left[\prod_{k<l}(\lambda_k-\lambda_l)^2\right]\delta\left(\sum_m\lambda_m\right)\exp\left(-\sum_n\lambda_n^2+z\lambda_1\right).\end{aligned} \tag{A6}$$

To simplify the exponents we shift the λ's so as to keep their sum equal to zero:

$$\lambda_1'=\lambda_1+\frac{z}{2N}-\frac{z}{2}$$

$$\lambda_i'=\lambda_i+\frac{z}{2N}\quad (i\neq 1), \tag{A7}$$

$$G(z;N)=N\int\frac{\Pi_id\lambda_i}{\mathcal{N}}\left[\prod_{1\neq k<l}(\lambda_k-\lambda_l)^2\right]\left[\prod_{k\neq 1}\left(\lambda_1+\frac{z}{2}-\lambda_k\right)^2\right]\delta\left(\sum_i\lambda_i\right)\exp\left(-\sum_i\lambda_i^2+z^2(N-1)/(4N)\right). \tag{A8}$$

The δ function can now be easily eliminated, using the exponential representation $\int d\alpha\ \exp(i\alpha\Sigma_i\lambda_i)$:

$$G(z;N)=Ne^{z^2(N-1)/(4N)}\int \frac{d\alpha\Pi_i d\lambda_i}{\mathcal{N}}\Bigg[\prod_{1\neq k<l}(\lambda_k-\lambda_l)^2\Bigg]\Bigg[\prod_{k\neq 1}\left(\lambda_1+\frac{z}{2}-\lambda_k\right)^2\Bigg]\exp\left(-\sum_i(\lambda_i-i\alpha/2)^2-\alpha^2N/2\right)$$
$$=Ne^{z^2(N-1)/(4N)}\int \frac{\Pi_i d\lambda_i}{\mathcal{N}}\Bigg[\prod_{1\neq k<l}(\lambda_k-\lambda_l)^2\Bigg]\Bigg[\prod_{k\neq 1}\left(\lambda_1+\frac{z}{2}-\lambda_k\right)^2\Bigg]\exp\left(-\sum_i\lambda_i^2\right). \tag{A9}$$

Let us isolate the integral over $\lambda_i(i\neq 1)$:

$$G(z;N)=Ne^{z^2(N-1)/(4N)}\int d\lambda_1 e^{-\lambda_1^2}e^{(\lambda_1+z/2)^2}\Bigg[\int \frac{\Pi_{i\neq 1}d\lambda_i'}{\mathcal{N}}\left(\prod_{k<l}(\lambda_k'-\lambda_l')^2\right)\exp\left(-\sum_i\lambda_i'^2\right)\Bigg]_{\lambda_1'=\lambda_1+z/2}. \tag{A10}$$

The integral in square brackets, involving the Vandermonde determinant $\Pi_{k<l}(\lambda_k'-\lambda_l')$, equals [24]

$$\frac{1}{N}\sum_{j=0}^{N-1}\phi_j^2\left(\lambda_1+\frac{z}{2}\right),\quad \phi_j(x)\equiv(2^j j!\sqrt{\pi})^{-1/2}e^{x^2/2}\left(-\frac{d}{dx}\right)^j e^{-x^2}. \tag{A11}$$

We thus obtain

$$G(z;N)=e^{z^2(N-1)/(4N)}\int d\lambda_1 e^{-\lambda_1^2}e^{(\lambda_1+z/2)^2}\sum_{j=0}^{N-1}\frac{1}{2^j j!\sqrt{\pi}}e^{(\lambda_1+z/2)^2}\Bigg[\left(-\frac{d}{d\lambda_1}\right)^j e^{-(\lambda_1+z/2)^2}\Bigg]^2$$
$$=e^{z^2(N-1)/(4N)}\int d\lambda_1 e^{-\lambda_1^2}\sum_{j=0}^{N-1}\frac{1}{2^j j!\sqrt{\pi}}\left[H_j\left(\lambda_1+\frac{z}{2}\right)\right]^2$$
$$=e^{z^2(N-1)/(4N)}\sum_{j=0}^{N-1}L_j^0\left(-\frac{z^2}{2}\right)$$
$$=e^{z^2(N-1)/(4N)}L_{N-1}^1\left(-\frac{z^2}{2}\right) \tag{A12}$$

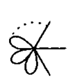
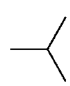
in terms of the Hermite (H_j) and Laguerre (L_β^α) polynomials.

APPENDIX B: PROOF OF Eq. (21)

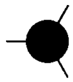
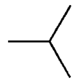
To prove Eq. (21), we must first evaluate the j-loop tadpole diagrams appearing in Eq. (20). Contracted legs come from $F_{x,\mu\nu}^{(1)}$, while external legs necessarily originate from $F_{x,\mu\nu}^{(1)}$ and $F_{x,\mu\nu}^{(2)}$. The corresponding vertex comes from

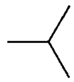
$$-\frac{1}{g_0^2}\sum_{\mu\nu}\mathrm{tr}[1-\exp(ig_0F_{x,\mu\nu})], \tag{B1}$$

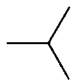
taking $2j+2$ powers from the exponent. We have

 = 
$$\cdot\frac{2!(ig_0)^{2j}}{(2j+2)!}\cdot\left[\frac{2j+2}{N^2-1}F(2j+2;N)\right]\cdot\left(\frac{1}{2}\right)^j \tag{B2}$$

The first factor above is the ratio of Taylor coefficients for the vertices on the left-hand side (LHS) and RHS; the factor in square brackets is the outcome of color contractions; the factor $(1/2)^j$ is the outcome of j one-loop integrations. Combining Eqs. (B2) and (20) we find

 $= $  $\cdot \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(ig_0)^n}{[2-2w(g_0)]^{n/2}} \frac{d^n}{dz^n} G'(z;N)|_{z=0} \right\} \cdot \frac{2[2-2w(g_0)]^{1/2}}{(ig_0)(N^2-1)}$

$=$  $\cdot G'\left(\frac{ig_0}{[2-2w(g_0)]^{1/2}};N\right) \cdot \frac{2[2-2w(g_0)]^{1/2}}{(ig_0)(N^2-1)}$

$=$  $\cdot [1-w(g_0)]$ (B3)

In the last equality, use was made of Eq. (17).

APPENDIX C: DRESSING THE FOUR-POINT VERTEX

The bare four-point vertex contains parts coming from $\mathrm{tr}\{F^{(1)}_{x,\mu\nu}F^{(1)}_{x,\mu\nu}F^{(1)}_{x,\mu\nu}F^{(1)}_{x,\mu\nu}\}$, $\mathrm{tr}\{F^{(2)}_{x,\mu\nu}F^{(2)}_{x,\mu\nu}\}$, and $\mathrm{tr}\{F^{(1)}_{x,\mu\nu}F^{(3)}_{x,\mu\nu}\}$. In general, these are expected to dress differently, thus yielding a dressed vertex which is not merely proportional to the bare one. In all other respects, this calculation is a direct generalization of the three-point vertex case.

We will not present the final expression for the dressed four-point vertex, since we will not be needing it in Sec. III; rather we evaluate the one new ingredient present in this case: the sum over all pairwise contractions of $\mathrm{tr}\{(F^{(1)}_{x,\mu\nu})^n\}$, with four legs left external. The result can be written as

$$\left(\frac{1}{2}\right)^{(n-4)/2} F^{(1)a}_{x,\mu\nu}F^{(1)b}_{x,\mu\nu}F^{(1)c}_{x,\mu\nu}F^{(1)d}_{x,\mu\nu}T^{abcd}, \tag{C1}$$

where T^{abcd} is necessarily of the form

$$T^{abcd} = \alpha\, \mathrm{tr}\{T^a(T^bT^cT^d + \text{permutations})\} + \beta(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}). \tag{C2}$$

We must compute α and β for generic n, N.

The tensors multiplying α and β above are in general independent, except for the cases $N=2$, $N=3$. One way to see this is by taking the scalar product of T^{abcd} (a real tensor) with itself:

$$\begin{aligned} T^{abcd}T^{abcd} &= 3(N^2-1)(N^2+1)\left[\alpha^2 \frac{N^4-6N^2+18}{8N^2(N^2+1)} + \alpha\beta \frac{2N^2-3}{N(N^2+1)} + \beta^2\right] \\ &= 3(N^2-1)(N^2+1)\left[\left(\beta + \frac{2N^2-3}{2N(N^2+1)}\alpha\right)^2 + \alpha^2 \frac{(N^2-9)(N^2-4)}{8(N^2+1)^2}\right]. \end{aligned} \tag{C3}$$

The above can vanish only for $N=2$ or $N=3$, if $\beta = -\frac{1}{4}\alpha$.

To compute α and β we further contract T^{abcd} with either $:\delta^{ef}\delta^{gh}:$ or $:\mathrm{tr}\{T^eT^fT^gT^h\}:$, to arrive at two relations for α and β:

$$\begin{aligned} &\frac{4!(N^2-1)}{2N}[(2N^2-3)\alpha + 2N(N^2+1)\beta] = n(n-2)F(n;N), \\ &\frac{4!(N^2-1)}{16N^2}[(N^4-6N^2+18)\alpha + 4N(2N^2-3)\beta] = \frac{n}{2}F(n+2;N) - \frac{n(2N^2-4+n)}{4N}F(n;N) \\ &\qquad + \frac{n(n-1)(N^2+n-3)}{8N^2}F(n-2;N). \end{aligned} \tag{C4}$$

The solution of these linear equations gives the required expressions for α and β.

For $N=2$, $N=3$, Eqs. (C4) are linearly dependent, as expected. Since the two tensors in T^{abcd} are proportional to each other in this case, we can set $\alpha=0$. Then, from Eq. (C4) we find

$$\alpha = 0, \quad \beta = \frac{n(n-2)F(n;N)}{4!(N^2-1)(N^2+1)} \quad (N=2 \text{ or } N=3). \tag{C5}$$

APPENDIX D: RESULTS FOR THE ADJOINT REPRESENTATION

We calculate

$$F_{\rm Adj}(n;N)=\frac{1}{2^{n/2}(n/2)!}\sum_{P\in S_n}\delta_{a_1a_2}\delta_{a_3a_4}\cdots\delta_{a_{n-1}a_n}{\rm tr}\{\mathcal{T}^{P(a_1)}\mathcal{T}^{P(a_2)}\cdots\mathcal{T}^{P(a_n)}\}. \tag{D1}$$

Here, $\mathcal{T}^a$ denote su(N) generators in the adjoint representation. We can relate them to the fundamental representation using the standard decomposition

$$(N)\otimes(N^*)\to(N^2-1)\oplus(1). \tag{D2}$$

In terms of the generators, this says that there exists a unitary matrix U such that

$$U^\dagger(T^a\otimes\mathbb{1}+\mathbb{1}\otimes T^{a*})U=\mathcal{T}^a\oplus(0). \tag{D3}$$

Using an $N^2\times(N^2-1)$ projector P [the $(N^2-1)\times(N^2-1)$ unit matrix augmented by a row of zeros],

$$\mathcal{T}^a=P^\dagger U^\dagger[T^a\otimes\mathbb{1}+\mathbb{1}\otimes(T^a)^*]UP. \tag{D4}$$

Substituting this in Eq. (D1) and making use of Eq. (A3) we find

$$F_{\rm Adj}(n;N)=\frac{1}{\mathcal{N}}\int\prod_a d\theta^a e^{-\theta^2/2}{\rm tr}\{(M\otimes\mathbb{1}+\mathbb{1}\otimes M^*)^n\}=\frac{1}{\mathcal{N}}\int\prod_a d\theta^a e^{-\theta^2/2}\sum_{m=0}^{n}\binom{n}{m}{\rm tr}\{M^m\}{\rm tr}\{M^{n-m}\}. \tag{D5}$$

The corresponding generating function is now given by

$$F_{\rm Adj}(n;N)=\frac{d^n}{dz^n}G_{\rm Adj}(z;N)\big|_{z=0},$$

$$G_{\rm Adj}(z;N)=\sum_{n_1,n_2=0}^{\infty}\frac{z^{n_1+n_2}}{n_1!n_2!}\frac{1}{\mathcal{N}}\int\prod_a d\theta^a e^{-\theta^2/2}{\rm tr}\{M^{n_1}\}{\rm tr}\{M^{n_2}\}$$
$$=\sum_{i,j}\int\frac{\Pi_m d\lambda_m}{\mathcal{N}}\left[\prod_{k<l}(\lambda_k-\lambda_l)^2\right]\delta\left(\sum_m\lambda_m\right)\exp\left(-\sum_n\lambda_n^2+z(\lambda_i+\lambda_j)\right). \tag{D6}$$

A somewhat tedious integration over λ_m, by analogy with Eqs. (A11), (A12), gives

$$G_{\rm Adj}(z;N)=G(2z;N)+e^{z^2(N-2)/(2N)}\left\{\left[L_{N-1}^1\left(-\frac{z^2}{2}\right)\right]^2-\sum_{n=0}^{N-1}\left[L_n^0\left(-\frac{z^2}{2}\right)\right]^2\right.$$
$$\left.-2\sum_{n=1}^{N-1}\sum_{m=0}^{N-1-n}\left(-\frac{z^2}{2}\right)^{n+2m}\frac{1}{m!(n+m)!}L_{N-n-m-1}^{n+2m+1}\left(-\frac{z^2}{2}\right)\right\}. \tag{D7}$$

From this point on, dressing the variant action propagator proceeds just as in Eqs. (8), (15), leading to

$$w_{\rm var}(g_0)=\left\{\sum_{n=0}^{\infty}\frac{1}{[2-2w_{\rm var}(g_0)]^{n/2}}\frac{(ig_0)^n}{n!}\left(\frac{\beta}{2N}F(n+1;N)+\frac{\beta_A}{2(N^2-1)}F_{\rm Adj}(n+1;N)\right)\right\}$$
$$\times\frac{2(ig_0)}{N^2-1}[2-2w_{\rm var}(g_0)]^{1/2}+1. \tag{D8}$$

Substituting the definitions of $G(z;N)$ and $G_{\rm Adj}(z;N)$ in the above immediately produces Eq. (33).

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